M2J Lecture 10/03/12

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Posing a linear system

- Suppose a sports stadium sells three products, 1) tickets, 2) drinks, 3) food. Suppose on a given day 55000 transactions are processed with total sales of $1,670,000 and total food / drink sales of 70,000. If each ticket costs $40, each drink $4, and each food item $6, how much of each was purchased.

Define Variables

- Suppose a sports stadium sells three products, 1) tickets, 2) drinks, 3) food. Suppose on a given day 55000 transactions are processed with total sales of $1,670,000 and total food / drink sales of 70,000. If each ticket costs $40, each drink $4, and each food item $6, how much of each was purchased.

  - Variables are the quantities you want to find.

$$x_1 = \text{number of tickets}$$

$$x_2 = \text{number of drinks}$$

$$x_3 = \text{number of food items}$$

Mathematically Formulate Information

- Suppose a sports stadium sells three products, 1) tickets, 2) drinks, 3) food. Suppose on a given day 55000 transactions are processed with total sales of $1,670,000 and total food / drink sales of 70,000. If each ticket costs $40, each drink $4, and each food item $6, how much of each was purchased.

  - 3 unknown variables requires three distinct pieces of information (i.e. equations).

$$\begin{aligned} x_1 + x_2 + x_3 &= 55000 \\ 40x_1 + 4x_2 + 6x_3 &= 1670000 \\ 4x_2 + 6x_3 &= 70000 \end{aligned}$$

Form Augmented Matrix

- Suppose a sports stadium sells three products, 1) tickets, 2) drinks, 3) food. Suppose on a given day 55000 transactions are processed with total sales of $1,670,000 and total food / drink sales of 70,000. If each ticket costs $40, each drink $4, and each food item $6, how much of each was purchased.

$$\begin{aligned} x_1 + x_2 + x_3 &= 55000 \\ 40x_1 + 4x_2 + 6x_3 &= 1670000 \\ 4x_2 + 6x_3 &= 70000 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 55000 \\ 40 & 4 & 6 & 1670000 \\ 0 & 4 & 6 & 70000 \end{array}\right]$$

Solve

- Suppose a sports stadium sells three products, 1) tickets, 2) drinks, 3) food. Suppose on a given day 55000 transactions are processed with total sales of $1,670,000 and total food / drink sales of 70,000. If each ticket costs $40, each drink $4, and each food item $6, how much of each was purchased.

$$x_1 = 40000, \qquad x_2 = 10000, \qquad x_3 = 5000$$

Terminology

- A matrix is said to be in ***row echelon form*** if:

1. The first nonzero entry in each row is 1(referred to as a **leading one**)
2. The entry below each leading one is 0.
3. Any rows containing all zeros are at the bottom.

Example

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \end{array}\right]$$

Why Row Echelon Form?

- Because this it is easy to compute a solution using back substitution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \longrightarrow \begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned}$$

Terminology

- ***Gaussian Elimination*** is the process of using elementary row operations to bring a matrix to row echelon form.
- Note: This is a very methodical process that can be applied almost the same way to any problem.

Gaussian Elimination

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

- Step 1: Turn top left corner into a "leading one".
  - .5 * R1

$$2 \quad 2 \quad 4 \quad 18 \rightarrow 1 \quad 1 \quad 2 \quad 9$$

Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

- Step 2: Use that "leading one" to eliminate all elements below it (turn them into 0).
  - -2*R1 + R2, Replace R2 with this
  - -3*R1 + R3, Replace R3 with this

Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$$-2R_1 + R_2$$

-2	-2	-4	-18
2	4	-3	1
0	2	-7	-17

- Step 2: Use that "leading one" to eliminate all elements below it (turn them into 0).
  - -2*R1 + R2, Replace R2 with this
  - -3*R1 + R3, Replace R3 with this

Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & \textcircled{2} & -7 & -17 \\ 0 & \textcircled{3} & -11 & -27 \end{array}\right]$$

- Step 3: Turn next diagonal element into a leading one : .5*R2
- Step 4: Eliminate all non-zeros below it.
  - -3*R2+R3

Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

- Step 5 - Turn the next diagonal element into a leading one.
  - -2*R3

General Process

$$\left[\begin{array}{cccc|c} 2 & 2 & 2 & 4 & 18 \\ -3 & 2 & 4 & -3 & 1 \\ 2 & 3 & 6 & -5 & 0 \\ -2 & 1 & 4 & -3 & 11 \end{array}\right]$$

- Turn first diagonal element into a leading one by multiplying the entire row by a constant (1/2).
- Eliminate all the zeros below it.
- Continue to next column until finished.

Possible Problems

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & -3 & 1 \\ 0 & 2 & 6 & 0 \end{array}\right]$$

- You cant turn a 0 into a 1 by multiplication.
- But, you can swap rows.

The Fix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & -3 & 1 \\ 0 & 2 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

- Swap rows 2 and three.
- Proceed as before.

Reduced Row Echelon Form

- A matrix is said to be in ***Reduced Row Echelon Form*** if

  1. It is in row echelon form.
  2. Each leading one is the only non-zero element in its column.

Examples

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 9 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Possible Solution Outcomes

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow \begin{array}{l} x = 3 \\ y = -2 \\ z = 5 \end{array}$$

- Unique Solution

Possible Solution Outcomes

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right] \longrightarrow \begin{array}{l} x = 3 \\ y = -2 \\ 0 = 5 \end{array}$$

- No Solution

Possible Solution Outcomes

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x = 3 \\ y = -2 \\ z = \text{anything} \end{array}$$

- Infinitely Many Solutions
- Final Row $0x + 0y + 0z = 0$

Possible Solution Outcomes

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x + y = 3 \\ y + z = -2 \end{array}$$

- Effectively, two equations and three unknowns.
  - Underdetermined.

Possible Solution Outcomes

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x = 3 - y \\ z = -2 - y \end{array}$$

- For each ‘y’, you get a solution.
  - Called a **one parameter family of solutions.**

Possible Solution Outcomes

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{aligned} x_1 - x_3 + 2x_4 &= 5 \\ x_2 + x_3 &= 2 \end{aligned}$$

- For each ‘y’, you get a solution.
  - Called a **one parameter family of solutions.**

Possible Solution Outcomes

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_1 = 5 + x_3 - 2x_4 \\ x_2 = 2 - x_3 \end{array}$$

- For each ‘x3, x4’, you get a solution.
  - Called a **two parameter family of solutions.**

Possible Solution Outcomes

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- These two have the exact same two parameter family of solutions.
- The initial number of equations (ie. rows) can be deceiving!